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Differentiable structures with zero entropy on simply connected 4-manifolds

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Abstract. We show that a closed 4-dimensional simply connected topological manifold M admits a differentiable structure with a C^∞ Riemannian metric whose geodesic flow has zero topological entropy if and only if M is homeomorphic to S^4, $\mathbb{CP}^2$, $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2 \# \mathbb{CP}^2$.

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1 Introduction

The Riemannian metric of constant Gaussian curvature one on S^2 and the flat metric on $\mathbb{T}^2$ have geodesic flows with zero topological entropy. On the other hand since the fundamental group of a closed orientable surface of genus ≥ 2 has exponential growth, it follows from a result of E. Dinaburg [4] that any Riemannian metric on a closed oriented surface of genus ≥ 2 will have a geodesic flow with positive topological entropy. Hence a closed orientable surface M admits a Riemannian metric whose geodesic flow has zero topological entropy if and only if M is diffeomorphic to S^2 or $\mathbb{T}^2$. Here we propose a version of this fact for closed simply connected 4-manifolds.

Let M be a closed topological manifold. We shall say that a differentiable structure on M has *zero entropy* if it admits a C^∞ Riemannian metric g such that the topological entropy $h_{top}(g)$ of the geodesic flow of g is zero. Our aim is to show:

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Theorem. *Suppose that M is 4-dimensional and simply connected. Then M admits a differentiable structure with zero entropy if and only if M is homeomorphic to S^4, $\mathbb{CP}^2$, $S^2 \times S^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2\#\mathbb{CP}^2$.*

Most of the work in the proof of the theorem consists in showing the existence of smooth Riemannian metrics on $\mathbb{CP}^2\#\mathbb{CP}^2$ with zero topological entropy. The metrics that we will use were first introduced by J. Cheeger in [3].

2 Rational homotopy and topological entropy

Let M^n be a closed connected and simply connected smooth n dimensional manifold.

The manifold M is said to be *rationally elliptic* if the total rational homotopy $\pi_*(M)\otimes\mathbb{Q}$ is finite dimensional, i.e. there exists a positive integer i_0 such that for all $i \geq i_0$, $\pi_i(M)\otimes\mathbb{Q} = 0$. The manifold M is said to be *rationally hyperbolic* if it is not rationally elliptic (cf. [6, 7, 11] and references therein). The next lemma is certainly well known and we include a proof for the sake of completeness.

Lemma 2.1. *Suppose that M is 4-dimensional and let b_2 be the second Betti number of M. If M is rationally elliptic then $b_2 \leq 2$.*

Proof. It is shown in [9, Corollary 1.3] (cf. also [5]) that if M^n is rationally elliptic then,

$$\sum_{k\geq 1} 2k \dim\ (\pi_{2k}(M)\otimes\mathbb{Q}) \leq n. \tag{1}$$

Since M is simply connected the Hurewicz isomorphism theorem implies that

$$b_2 = \dim\ H_2(M,\mathbb{Q}) = \dim\ (\pi_2(M)\otimes\mathbb{Q}).$$

Since $n = 4$, using (1) we obtain $2\,b_2 \leq 4$. □

The next lemma was probably known to some experts but we have not not been able to find it in the literature and so we include a proof of it.

Lemma 2.2. *Let M be a closed smooth simply connected 4-manifold. Then M is rationally elliptic if and only if M is homeomorphic to S^4, $\mathbb{CP}^2$, $S^2 \times S^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2\#\mathbb{CP}^2$.*

Proof. Suppose that M is rationally elliptic. By Lemma 2.1, $b_2 \leq 2$. Since M is smooth, the Kirby-Siebenmann obstruction vanishes. Therefore by M. Freedman's theory [8], the homeomorphism type of M is completely determined by the intersection form of M. It follows that if $b_2 = 0$, M is homeomorphic to S^4 and if $b_2 = 1$, M is homeomorphic to $\mathbb{CP}^2$. When $b_2 = 2$, the possible intersection forms are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These forms correspond to $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ respectively.

On the other hand S^4, $\mathbb{CP}^2$ and $S^2 \times S^2$ are homogeneous spaces and hence they are rationally elliptic [17]. In [7] it is shown that Poincaré complexes M such that $H^*(M, \mathbb{Q})$ is generated by two elements are rationally elliptic, hence $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ are rationally elliptic. □

We now recall the following result essentially pointed out by M. Gromov in [10, Section 2.7]. A proof can be found in [13, 15]. Related results appear in [1].

Theorem 2.3. *Let M be a closed smooth simply connected rationally hyperbolic manifold. Then for any C^∞ Riemannian metric g on M, $h_{top}(g) > 0$.*

If we combine this theorem with Lemma 2.2 we obtain right away:

Corollary 2.4. *Let M be a closed simply connected 4-dimensional topological manifold. If M admits a differentiable structure with zero entropy, then M is homeomorphic to one of the five manifolds listed in the theorem in the introduction.*

In [1] I. Babenko gives a lower bound for $h_{top}(g)$ in terms of b_2 and other geometric quantities. It was this result of Babenko that motivated the theorem in the introduction.

3 A smooth Riemannian metric on $\mathbb{CP}^2 \# \mathbb{CP}^2$ whose geodesic flow has zero topological entropy

On account of Corollary 2.4 to prove the theorem in the introduction it suffices to show that if M is homeomorphic to one of the five manifolds listed in the theorem, then M admits a differentiable structure with zero entropy. We shall endow each of the five manifolds with their canonical smooth structures.

We shall use the following simple lemma whose proof we omit.

Lemma 3.1.

1. *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. Endow $M_1 \times M_2$ with the product metric $g_1 \times g_2$. Then*

$$h_{top}(g_1 \times g_2) = \sqrt{[h_{top}(g_1)]^2 + [h_{top}(g_2)]^2}.$$

2. *Let* $(M, g_M) \mapsto (N, g_N)$ *be a Riemannian submersion where* M *and* N *are compact manifolds. Then* $h_{top}(g_M) \geq h_{top}(g_N)$.

The standard symmetric metrics on S^4 and $\mathbb{CP}^2$ have all the geodesics closed and with the same period, and hence their geodesic flows have zero topological entropy. On $S^2 \times S^2$ consider the product metric of the round metric on S^2; it follows from part (1) in Lemma 3.1 that the geodesic flow of the product metric has zero entropy.

The manifold $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ is the non-trivial S^2-bundle over S^2 and it is known to be diffeomorphic to the space that we now describe. Represent $S^3 \subset \mathbb{C}^2$ as pairs of complex numbers (z_1, z_2) with $|z_1|^2 + |z_2|^2 = 1$. Let S^1 act on S^3 by

$$(w, (z_1, z_2)) \mapsto (wz_1, wz_2),$$

where $w \in S^1$ is a complex number with modulus one. Let S^1 also act on S^2 by rotations. Consider the space $M = S^3 \times_{S^1} S^2$ obtained by taking the quotient of $S^3 \times S^2$ by the diagonal action of S^1. The manifold M is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Endow S^3 and S^2 with the canonical metrics of curvature one. By part (1) of Lemma 3.1 the product metric on $S^3 \times S^2$ has zero entropy. By part (2) in Lemma 3.1 the submersion metric on $M = S^3 \times_{S^1} S^2$ will also have a geodesic flow with zero entropy.

We are left with the case of $M = \mathbb{CP}^2 \# \mathbb{CP}^2$. The manifold M can be obtained from two copies of $S^3 \times_{S^1} D^2$ where D^2 is the 2-disk and S^1 acts diagonally, glued along their boundary $S^3 \times_{S^1} S^1 = S^3$ by an orientation reversing map. The metrics that we will use were already considered by J. Cheeger in [3]. Let us describe them.

Denote by g_t the metric on S^3 which is obtained from the canonical metric of curvature one by multiplying with t^2 ($t \neq 0$) in the directions tangent to the S^1-orbits. The restriction to S^3 of the linear action of $SU(2)$ on $\mathbb{C}^2$ is by isometries and commutes with the S^1-action. Hence the group $G := SU(2) \times S^1$ acts on (S^3, g_t) by isometries. It is known that (S^3, g_t) can be viewed as distance spheres on $\mathbb{CP}^2$ with the metric induced by the Fubini-Study metric. For $t^2 \leq 1$ they are called Berger spheres. We refer to [19] for details.

Now equip $\mathbb{R}^2$ with a metric $h_t (t^2 > 1)$ given in polar coordinates by :

$$h_t(\partial/\partial r, \partial/\partial r) = 1 \quad h_t(\partial/\partial r, \partial/\partial \theta) = 0 \quad h_t(\partial/\partial \theta, \partial/\partial \theta) = f_t^2(r)$$

where $f_t(r)$ is a positive smooth function with the properties $f_t(0) = 0$, $f_t'(0) = 1$ and $f_t(r) \equiv 2\pi t^2/\sqrt{t^4 - 1}$ for sufficiently big $r > R$.

Set $\eta = S^3 \times_{S^1} \mathbb{R}^2$ with the submersion metric. If we restrict to the disk bundle $D_{\bar{R}}(\eta)$ with $\bar{R} > R$, then an annular neighborhood of the boundary splits isometrically as $\partial D_{\bar{R}}(\eta) \times I$ where I denotes an interval. In fact, $A = \{X \in \mathbb{R}^2 \mid R < \| X \| < \bar{R}\}$ splits isometrically as $S^1 \times I$ and S^1 acts trivially on I. Then

$$S^3 \times_{S^1} A = S^3 \times_{S^1} (S^1 \times I) = (S^3 \times_{S^1} S^1) \times I = S^3 \times I$$

and a calculation shows that $S^3 = \partial D_{\bar{R}}(\eta)$ gets back the metric g_1 of constant curvature one (cf. [3]). Since the metric splits as a product $S^3 \times I$ near the boundary, by glueing two such disk bundles we get a smooth metric on $\mathbb{CP}^2\#\mathbb{CP}^2$ that we denote by g_{Ch} and we call the *Cheeger metric*. The orientation reversing glueing map on the boundary S^3 that we shall use is the reflection

$$(z_1, z_2) \mapsto (\bar{z}_1, z_2).$$

A Hamiltonian H on a symplectic manifold X^{2n} is said to be *completely integrable with periodic integrals* if there exists a Hamiltonian action of the $n-1$ dimensional torus $\mathbb{T}^{n-1}$ on X with principal orbits of dimension $n-1$ and such that it leaves H invariant.

Proposition 3.2. *The Hamiltonian $H_{Ch} : T^*\left(\mathbb{CP}^2\#\mathbb{CP}^2\right) \to \mathbb{R}$ that generates the geodesic flow of the Cheeger metric g_{Ch} on $\mathbb{CP}^2\#\mathbb{CP}^2$ is completely integrable with periodic integrals.*

We remark that in [16] we constructed completely integrable geodesic flows on $\mathbb{CP}^n\#\mathbb{CP}^n$ but *only* for *n odd* and the integrals were not necessarily periodic.

Before proving the proposition we recall Theorem 3.1 in [13] (for a non commutative version of the theorem see [14]):

Theorem 3.3. *Let H be a Hamiltonian on a symplectic manifold X and let N be a compact regular energy level of H. Then if H is completely integrable with periodic integrals, the Hamiltonian flow of H restricted to N has zero topological entropy.*

From Proposition 3.2 and Theorem 3.3 we derive the following corollary thus concluding the proof of the theorem in the introduction.

Corollary 3.4. *The Cheeger metric g_{Ch} on $\mathbb{CP}^2\#\mathbb{CP}^2$ has $h_{top}(g_{Ch}) = 0$.*

We would like to point out that it is not sufficient to show that the geodesic flow of a Riemannian metric g is completely integrable to obtain that $h_{top}(g) = 0$ as it is shown by the recent remarkable counterexample of Bolsinov and Taimanov [2]. One needs the first integrals to be "nice enough", like the periodic integrals in Theorem 3.3.

Proof of Proposition 3.2. Recall that the group $G = SU(2) \times S^1$ acts on S^3 as follows. Let (z_1, z_2) be a point in S^3 and let $(U, w) \in G$ where $U \in SU(2)$ and $w \in S^1$ is a complex number with modulus one. Then the action is given by

$$((U, w), (z_1, z_2)) \mapsto U(wz_1, wz_2).$$

The group G contains a two torus $\mathbb{T}^2$ that acts on S^3 as follows. If $(w_1, w_2) \in \mathbb{T}^2$ where w_1 and w_2 are complex numbers with modulus one, then

$$((w_1, w_2), (z_1, z_2)) \mapsto (w_1 z_1, w_2 z_2).$$

Let us denote this action by $\rho(w_1, w_2)$. Let $r : S^3 \to S^3$ be the reflection

$$r(z_1, z_2) = (\bar{z}_1, z_2).$$

One easily checks that

$$r \circ \rho(w_1, w_2) = \rho(\bar{w}_1, w_2) \circ r. \tag{2}$$

The action ρ of $\mathbb{T}^2$ on S^3 naturally extends to $S^3 \times \mathbb{R}^2$ and since it commutes with the diagonal S^1-action it descends to an action on the disk bundle $D_{\bar{R}}(\eta)$. One can easily check that on the boundary of $D_{\bar{R}}(\eta)$ we recover the action ρ of $\mathbb{T}^2$ on S^3.

Let D_1 and D_2 be two copies of $D_{\bar{R}}(\eta)$. We let $\mathbb{T}^2$ act on D_1 by $\rho(w_1, w_2)$ and on D_2 by $\rho(\bar{w}_1, w_2)$. Using (2) we see that we can glue these two actions to obtain an action of $\mathbb{T}^2$ on $D_1 \cup_r D_2 = \mathbb{CP}^2 \# \mathbb{CP}^2$. By construction this action is by isometries of the Cheeger metric.

To prove the proposition we need to find an extra circle action commuting with $\mathbb{T}^2$ and leaving the Hamiltonian of the Cheeger metric invariant. We need first some preliminaries.

Let X be a symplectic space with a Hamiltonian action of a Lie group G. Such an action is called *multiplicity free* if the algebra of the G-invariant functions on X is commutative under the Poisson bracket [12, p. 361]. It is known that the lift of the action of $G = SU(2) \times S^1$ on S^3 to T^*S^3 is multiplicity free [18]. Hence, if $H_t : T^*S^3 \to \mathbb{R}$ is the Hamiltonian of the metric g_t, then for any t and s, H_t and H_s Poisson commute. Note that the Hamiltonian flow of H_1, which corresponds to the metric of constant curvature one, has all the orbits closed and hence it generates a circle action on T^*S^3. Hence, $H_1 : T^*S^3 \to \mathbb{R}$ is a first integral of the geodesic flow of g_t whose Hamiltonian flow generates a circle action. The function H_1 naturally extends to $T^*(S^3 \times \mathbb{R}^2) = T^*S^3 \times T^*\mathbb{R}^2$ and since it is invariant under the lift of the diagonal action to $T^*(S^3 \times \mathbb{R}^2)$ it descends to $\Phi^{-1}(0)/S^1 = T^*\eta$ where Φ is the moment map of the lift of the S^1-action. Let $\widetilde{H}_1 : T^*\eta \to \mathbb{R}$ be the induced function. As before let D_1 and D_2 be two copies of the disk bundle $D_{\tilde{R}}(\eta)$. Note that an annular neighborhood of the boundary of $T^*D_{\tilde{R}}(\eta)$ splits as $T^*S^3 \times T^*I$. The function $\widetilde{H}_1$ is invariant under derivatives of translations on I. Therefore it will give rise to a smooth function on the cotangent bundle of $D_1 \cup_r D_2$ if it happens to be invariant under the map $(dr)^*$. Fix a point $x \in I$. One can easily see that the restriction of $\widetilde{H}_1$ to $T^*S^3 \times \{(x, 0)\}$ gives back the function H_1 which we know to be invariant under $(dr)^*$. Hence $\widetilde{H}_1$ extends to a smooth function on $T^*\left(\mathbb{CP}^2\#\mathbb{CP}^2\right)$ which is a first integral of the geodesic flow of the Cheeger metric g_{Ch} and whose Hamiltonian flow generates a circle action. Finally, by construction H_1 is invariant under the lift of the $\mathbb{T}^2$ action.

It only remains to check that the action of the 3-tours $\mathbb{T}^3$ on $T^*\left(\mathbb{CP}^2\#\mathbb{CP}^2\right)$ thus obtained has 3-dimensional orbits. For this take a point $(z_1, z_2) \in S^3$ such that the orbit of the action ρ of $\mathbb{T}^2$ on S^2 is 2-dimensional. Recall that the action ρ lifts to T^*S^3. Let $p \in T^*_{(z_1,z_2)}S^3$ be such that the closed orbit of the Hamiltonian flow of H_1 through p is not the orbit of a 1-parameter subgroup of $\mathbb{T}^2$. A generic p will have this property. Then the orbit of a point $(p, (x, 0)) \in T^*S^3 \times T^*I$ under $\mathbb{T}^3$ will be 3-dimensional. □

Remark 3.5. It is well known that the Riemannian metrics we considered in S^4, $\mathbb{CP}^2$ and $S^2 \times S^2$ have completely integrable geodesic flows. In [16] we described a large class of metrics with completely integrable geodesic flows on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. In fact, if we glue the two disk bundles D_1 and D_2 with the identity map, the proof of Proposition 3.2 shows that the Cheeger metrics thus obtained on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ also have completely integrable geodesic flows with periodic integrals. Hence the five manifolds listed in the theorem admit Riemannian metrics with completely integrable geodesic flows and nice first integrals.

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